

Multi-Particle Quantum Dynamics under Continuous Observation

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We formulate the dynamics of quantum many-body systems under real-time spatially-resolved measurement, and show that, in the continuous measurement limit, the indistinguishability of identical particles results in the complete suppression of relative positional decoherence. As a consequence, the indistinguishability persists in the course of quantum dynamics under weak continuous observation. We numerically demonstrate this result for ultracold atoms in an optical lattice. Our results can be applied to quantum feedback control of quantum many-body systems which may be realized in subwavelength spacing lattice systems.

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Indistinguishability of identical particles and measurement back-action are two fundamental postulates of quantum mechanics. The indistinguishability introduces multi-particle quantum interference and results in rich varieties of intriguing phenomena such as the Hong-Ou-Mandel effect [1] and Bose-Einstein condensation [2, 3]. On the other hand, the measurement back-action dramatically alters the dynamics of quantum systems so that the quantum state reduces to a particular measurement basis consistent with the outcome of a measurement process [4, 5]. However, a clear understanding of how the interplay between the indistinguishability and the measurement back-action affects the quantum many-body dynamics, in particular, under continuous observation, remains elusive.

Experimental techniques in ultracold atoms are now approaching the level at which such a fundamental question can be addressed. Recent realizations of single-site resolved detection [6–11] and addressing [12, 13] of atoms in an optical lattice have provided a powerful tool to investigate the quantum many-body dynamics at the level of single particles [14–19]. In particular, strongly correlated quantum walk of identical particles have been observed [13]. Further developments of those *in-situ* imaging techniques will allow us to perform a minimally destructive monitoring of multi-particle quantum dynamics [20–24]. This type of monitoring is an essential step to generalize the concepts in measurement-based quantum feedback control [25, 26] to quantum many-body systems and will provide new resources to quantum information processing. Furthermore, such a monitoring may offer an opportunity to investigate a novel effect of measurement back-action on quantum critical phenomena [27, 28]. Against such a backdrop, it is urgent to develop a theoretical tool to analyze the quantum many-body dynamics under a real-time observation and uncover the nature of the quantum indistinguishability in such a continuously monitored system.

In this Letter, we make progress toward achieving this goal. We consider a quantum dynamics of multiple iden-

tical atoms in an optical lattice subject to a spatially-resolved measurement. We formulate a theoretical description of the selective multi-particle quantum dynamics conditioned on measurement outcomes, and obtain a stochastic many-body Schrödinger equation in the weak and frequent measurement limit. Such a derivation of the continuous monitoring limit for identical particles has long remained an outstanding issue [29]. Remarkably, in this limit, we show that the indistinguishability of signals results in the complete suppression of the relative positional decoherence, which makes a striking contrast with the case of distinguishable particles. As a result, particle indistinguishability persists in the course of quantum dynamics even in the presence of the continuous observation. We demonstrate this result by numerical simulations of correlated quantum walks. In previous works concerning the site-resolved position measurement [30–34] and continuous position measurement of a single quantum particle [35–39], the quantum indistinguishability could not play such a nontrivial role.

Besides the fundamental interest, our results also have a practical importance in opening up the possibility of measurement-based feedback control of quantum many-body systems. In particular, our theoretical description and finding may have an important application to recently proposed subwavelength-spacing lattices [40–43], as discussed below.

We consider quantum position measurement of N identical atoms trapped in a lattice. While we focus on lattice systems in this Letter, the following formulation can be generalized to continuous space. Let $\hat{M}(X)$ be the measurement operator of atomic positions where X denotes the measurement outcome. When we consider photodetection as a measurement process, the quantum jump process associated with measurement back-action can be modeled by the Poisson stochastic process [44]. We describe the quantum dynamics of identical atoms subject to a spatially-resolved measurement by the fol-

lowing stochastic Schrödinger equation:

$$d|\psi\rangle = -\frac{i}{\hbar}\hat{H}|\psi\rangle dt - \frac{1}{2} \int dX \left(\hat{M}^\dagger(X)\hat{M}(X) - \langle \hat{M}^\dagger(X)\hat{M}(X) \rangle \right) |\psi\rangle dt + \int dX \left(\frac{\hat{M}(X)|\psi\rangle}{\sqrt{\langle \hat{M}^\dagger(X)\hat{M}(X) \rangle}} - |\psi\rangle \right) dN(X). \quad (1)$$

The first term on the right-hand side describes a unitary evolution under the Bose-Hubbard Hamiltonian $\hat{H} = -J \sum_m (\hat{b}_m^\dagger \hat{b}_{m+1} + \text{H.c.}) + (U/2) \sum_m \hat{n}_m(\hat{n}_m - 1)$. The other terms represent a non-unitary time evolution caused by the measurement back-action. In the no-count process in which no photons are detected, the time evolution of the quantum state is modified by the second term. For a single-particle case, this evolution is equivalent to that of the original coherent evolution because the non-Hermitian term $-(i/2) \int dX \hat{M}^\dagger(X)\hat{M}(X)$ is proportional to the identity operator and the contribution of the non-Hermitian term reduces merely to a global constant factor. In contrast, for multiple indistinguishable particles, the time evolution during the no-count process is qualitatively different because quantum interference between indistinguishable particles changes the measurement rate depending on atomic configurations. The third line in Eq. (1) describes the quantum jump process associated with photodetections where $dN(X)$ is Poisson stochastic process satisfying $dN(X)dN(Y) = \delta(X-Y)dN(X)$ and $E[dN(X)] = \langle \hat{M}^\dagger(X)\hat{M}(X) \rangle dt$. We denote $E[\dots]$ as an ensemble average over all measurement outcomes. We consider a general position measurement operator $\hat{M}(X) = \sqrt{\gamma} \sum_m f(X - md) \hat{b}_m^\dagger \hat{b}_m$, where γ is the parameter characterizing the measurement rate, d is the lattice constant, and f is an amplitude of a point spread function. We introduce an effective spatial resolution σ of optics in terms of the expansion coefficient of the displaced integration of f [45]. The dependence on the point spread function is solely described by this parameter.

Let us first consider N distinguishable particles whose position operators are given by $\hat{x}_i = \sum_{m=1}^N m|m\rangle_{ii}\langle m|$ where i denotes the label of each particle. In this case, we can obtain the continuous limit by generalizing results for a single particle model [36, 46]:

$$d\hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] dt - \frac{\gamma_{\text{tot}} d^2}{4\sigma^2} [\hat{X}_{CM}, [\hat{X}_{CM}, \hat{\rho}]] dt - \frac{d^2}{4\sigma^2} \sum_{i=1}^N \gamma_i [\hat{r}_i, [\hat{r}_i, \hat{\rho}]] dt + \sqrt{\frac{\gamma_{\text{tot}} d^2}{2\sigma^2}} \{ \hat{X}_{CM} - \langle \hat{X}_{CM} \rangle, \hat{\rho} \} dW + \sum_{i=1}^N \sqrt{\frac{\gamma_i d^2}{2\sigma^2}} \{ \hat{r}_i - \langle \hat{r}_i \rangle, \hat{\rho} \} dW_i, \quad (2)$$

where $\hat{X}_{CM} = \sum_{i=1}^N \hat{x}_i/N$ is the center-of-mass coordinate, $\hat{r}_i = \hat{x}_i - \hat{X}_{CM}$ is the relative position, $\gamma_{\text{tot}} = \sum_{i=1}^N \gamma_i$ is the total measurement rate, dW_i 's are independent Wiener processes satisfying $E[dW_i] = 0$ and $dW_i dW_j = \delta_{i,j} dt$, and we define $dW \equiv \sum_i \sqrt{\gamma_i/\gamma_{\text{tot}}} dW_i$. Here the Wiener process dW_i represents fluctuations of measurable signals coming from the i -th particle. Such distinguishable signals may be realized by utilizing, for example, state-selective imaging [12, 16–19, 23] or the degree of polarization [47]. Note that the measurement back-action contributes from the sum of the center-of-mass part and the relative positional part.

Let us next consider the continuous measurement limit of indistinguishable particles. To do so, we need to take the limit of both frequent ($\gamma \gg J/\hbar$) and weak ($\sigma \gg d$) measurements while keeping the ratio γ/σ^2 constant [35–39]. Otherwise, the unitary part of the evolution is completely suppressed due to the quantum Zeno effect [22]. In the continuous limit, the number of photodetections become large and it follows from the central limit theorem that we can model the stochastic process as $dN(X) \simeq \langle \hat{M}^\dagger(X)\hat{M}(X) \rangle dt + \sqrt{\langle \hat{M}^\dagger(X)\hat{M}(X) \rangle} dW(X)$ where $dW(X)$ is a Wiener stochastic process satisfying $dW(X)dW(Y) = \delta(X-Y)dt$ and $E[dW(X)] = 0$. Then, Eq. (1) separates into two parts; the deterministic part, which describes the average behavior of the density matrix, and the stochastic part which represents back-action caused by the uncertainty of measurement outcomes. Because of the particle number conservation, we can show that the former part becomes double commutation of the center-of-mass operator in the continuous limit. As for the latter part, a careful treatment about simultaneously taking the continuous limit and the infinite-range limit of the integral is required [45]. In this way, we obtain the stochastic Schrödinger equation of indistinguishable particles under continuous monitoring:

$$d\hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] dt - \frac{N^2 \gamma d^2}{4\sigma^2} [\hat{X}_{CM}, [\hat{X}_{CM}, \hat{\rho}]] dt + \sqrt{\frac{N^2 \gamma d^2}{2\sigma^2}} \{ \hat{X}_{CM} - \langle \hat{X}_{CM} \rangle, \hat{\rho} \} dW, \quad (3)$$

where $\hat{X}_{CM} = \sum_m m \hat{n}_m/N$. Remarkably, in contrast to distinguishable particles, the relative positional term is absent. The crucial point is that the indistinguishability of measurable signals introduces the multi-particle quantum interference between particles occupied in different sites and this results in the cancellation of the relative positional decoherence. The quantum indistinguishability could not play such a non-trivial role for the case of the site-resolved measurement [30–34]. Equation (S10) sets an extension of a model of continuous position measurement [35–39] to quantum many-body systems. Due to the difficulty of applying the original derivations [35–39] of the single particle model to indistinguishable multi-particle systems, a continuous monitoring model for iden-

tical particles has long remained elusive [29]. Note that we do *not* assume that the degrees of freedom of relative positions are frozen unlike in a rigid system [46, 48, 49].

Since relevant decoherence operators now reduce to a single observable, i.e., the center-of-mass coordinate, we can generalize the theory of quantum feedback control [25, 26], which is originally developed in homodyne detection in quantum optics, to quantum many-body systems. The center-of-mass position should be controllable by, for example, moving the global position of an optical lattice.

To discuss a physical consequence of the absence of the relative positional decoherence, let us focus on the decoherence rate of the off-diagonal term $\langle\{n_m\}|\hat{\rho}|\{n'_m\}\rangle$ of the density matrix:

$$\Gamma_{\{n_m\},\{n'_m\}} = \frac{N^2\gamma d^2}{4\sigma^2} (X_{CM} - X'_{CM})^2, \quad (4)$$

where $\{n_m\}$ denotes the Fock state and X_{CM} (X'_{CM}) is the center-of-mass coordinate of the state $\{n_m\}$ ($\{n'_m\}$). From Eq. (4), we can infer that there are in general three regimes in time evolution of multiple quantum dynamics under continuous monitoring:

(i) *Center-of-mass collapse regime.* A superposition of different center-of-mass states rapidly decoheres. The many-body wavefunction collapses into a state whose center-of-mass coordinate takes a well-localized value. This collapse occurs in the timescale of about $4\sigma^2/(N^2\gamma d^2 L^2)$, where L denotes a typical distance between the center-of-mass positions of the superposed Fock states.

(ii) *Inertial regime.* While the center-of-mass coordinate is well localized, because of the absence of the relative positional decoherence, the coherence within the subspace of the Fock states that take on the values close to the center of mass is preserved. As a result, the relative quantum motion of multiple indistinguishable particles is largely unaffected by continuous monitoring.

(iii) *Diffusive regime.* In the long time limit ($t \gg 4\sigma^2/(\gamma d^2)$), the coherence between the nearest Fock states whose center-of-mass coordinates differ by the minimal distance $\delta X_{CM} = 1/N$ is eventually lost, and particles undergo random walk. While the eventual advent of such diffusive transport can also be seen for distinguishable particles, the characteristic behavior such as the diffusion constant is qualitatively different depending on particle species because the coherence between the same center-of-mass Fock states survives in the course of measurement.

To illustrate these general properties, we perform numerical simulations. A particularly interesting and important regime is the second regime where coherent quantum dynamics of relative motion persists under continuous observation owing to the absence of relative positional decoherence. Such robustness can be captured by quantum walks of two particles. Later, we set $U = 0$ and

focus on quantum behavior caused by purely quantum statistics and measurement indistinguishability of identical particles. As long as we consider non-interacting particles, qualitatively the same argument will apply to a larger number of particles. To make a fair comparison, we perform numerical simulations under condition $\Gamma \equiv N^2\gamma d^2/\sigma^2 = \gamma_{\text{tot}}d^2/\sigma^2$ in which the number of measured resources is the same regardless of particle distinguishability. We consider the initial condition in which two particles are localized at adjacent sites. First, without measurement back-action, the density profile is almost independent of quantum statistics of the particles involved (Fig. 1a) [50]. In this case, the peaks of atom density spread in two straight directions, reflecting ballistic transport of atoms on lattice $\langle x^2 \rangle = 2J^2t^2/\hbar^2$ [51]. In contrast, if we perform continuous monitoring, the quantum dynamics appreciably changes depending on quantum statistics. While distinguishable particles exhibit uncorrelated random walks (Fig. 1b), indistinguishable particles exhibit ballistic and strongly-correlated walks (Fig. 1c, d). Ballistic transport implies that coherence between Fock states within a well-localized center-of-mass subspace is preserved, which is a characteristic feature of the inertial regime. This results from the fact that indistinguishability of measurable signals protect the quantum system from relative positional decoherence. The strong correlation arises from the multi-particle interference between quantum amplitudes

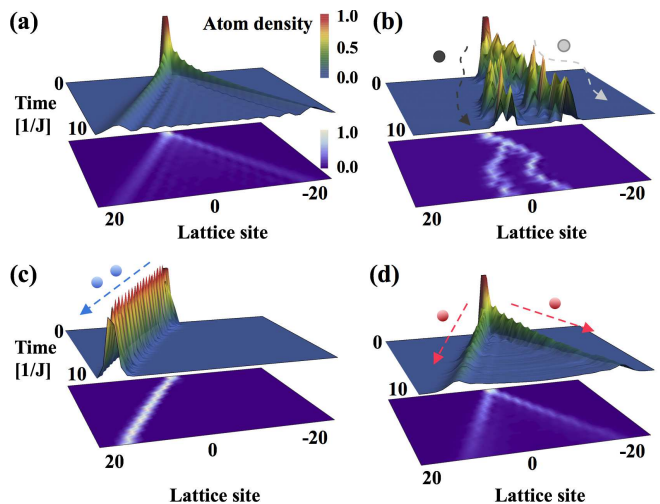


FIG. 1. (color online). Real-time quantum dynamics of two particles under continuous observation. (a) Unitary time evolution. Without measurement back-action, quantum statistics hardly affect the atom density. (b-d) Quantum dynamics under continuous monitoring for (b) distinguishable particles, (c) bosons, and (d) fermions, calculated for $\Gamma = 2.0$. Owing to the absence of the relative positional decoherence, (c) bunched and (d) anti-correlated quantum walks of indistinguishable particles persist, whereas distinguishable particles eventually exhibit uncorrelated random walks (b).

of identical particles. For bosons (fermions), two atoms move in the same (opposite) direction(s) due to constructive (destructive) interference of identical atoms. Because of rapid collapse of the center-of-mass coordinate, two bosons form a localized wave packet. The collapse occurs in the time scale of about $t_{\text{col}} \simeq (3\sqrt{2}\hbar^2/(\Gamma J^2))^{1/3}$. In contrast, for fermions, since the center-of-mass coordinate takes a localized value around zero due to the anti-correlation, continuous monitoring does not appreciably alter the quantum dynamics compared with the coherence dynamics, making striking contrast with the site-resolved measurement [31].

To discuss the diffusive regime, we numerically simulate the dynamics for a larger value of measurement strength Γ . Figure 2 (a) plots the square σ_r^2 of relative distance between two particles averaged over many realizations. For distinguishable particles, relative motion eventually exhibits random-walk behavior with the diffusion constant $D_c = 16J^2\sigma^2/(\gamma\hbar^2d^2)$ [52]. In contrast, quantum statistics serves as an effective attractive (repulsive) interaction for bosons (fermions), resulting in a diffusion constant smaller (larger) than D_c . Figure 2 (b) and (c) show typical realizations of bosons and fermions, respectively. The crucial point is that the coherence between Fock states that have the same center-of-mass value still remains nonvanishing owing to the complete suppression of decoherence, and quantum interference between indistinguishable particles can affect the diffusive dynamics.

In the opposite high-resolved limit $\sigma \ll d$, after taking the ensemble average over measurement outcomes,

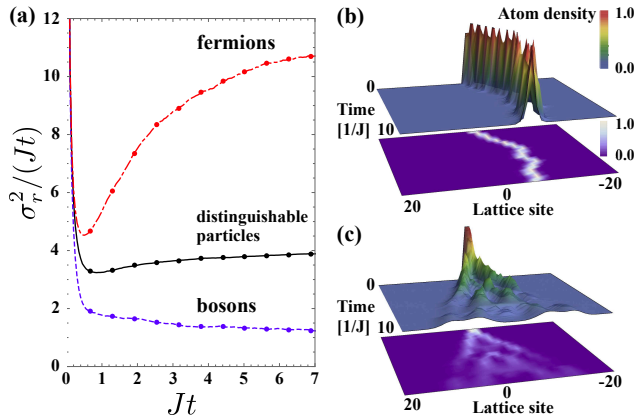


FIG. 2. (color online). Diffusive regime of two-particle quantum dynamics under continuous observation. (a) Variance of the relative distance divided by the elapsed time. Distinguishable particles (solid curve) show exponential convergence to $2D_c/J$. Bosons (blue dashed) and fermions (red dash-dotted) exhibit a stronger diffusion and a weaker diffusion, respectively. A typical realization of the atom density is shown for (b) bosons and (c) fermions. The results are calculated for initial condition of two adjacent particles with $\Gamma = 16.0$ and averaged over 10^3 realizations.

Eq. (1) reproduces the Lindblad type equation: $E[\dot{\rho}] = -(i/\hbar) [\hat{H}, \hat{\rho}] - (\gamma/2) \sum_m [\hat{n}_m, [\hat{n}_m, E[\hat{\rho}]]]$. This equation has been considered in the study of quantum dynamics under the site-resolved measurement [30, 32–34]. Without interaction $U = 0$, the coherence is rapidly lost during the time scale of $\sim 1/\gamma$ and particles exhibit diffusive behavior characteristic of classical random walk [53–56]. We note that, with a finite interaction $U \neq 0$, an exotic behavior such as anomalous diffusion can be found [32, 33]. Exploring such a non-trivial role of the interaction in the continuous measurement regime merits a further study.

While we assume an ideal situation of a unit collection efficiency of signals, our theory can be easily generalized so as to take into account uncollected signals by using the standard treatment of open quantum systems [4, 57]. For a possible effect of heating, although such effects can usually be made negligible by, for example, conducting an experiment in the deep Lamb-Dicke regime [58] and using Raman sideband cooling [9, 10, 21, 22], continuous imaging may cause heating of trapped atoms in the long time regime. In such a case, the theory should be modified by including higher bands [30]. To achieve a continuous monitoring, one needs to perform a low-resolved imaging of optical lattice systems. While the resolution σ can be changed by adjusting a numerical aperture of lens [59], such control also affects the collection efficiency of signals and may lead to a higher heating rate of atoms. To meet the requirement, we suggest that recently proposed sub-wavelength lattices [40–43] will offer promising systems for implementing continuous imaging because, in those systems, the lattice constant can be made much shorter than that of optical lattice systems, and both a low spatial resolution and a high collection efficiency of photons can be attained at the same time. Another promising direction is to use imaging light whose wavelength is longer than the lattice constant in which a heating effect can be substantially suppressed. Realizing such a minimally destructive observation will open up the possibility of measurement-based control of quantum many-body systems.

In summary, we have constructed a theoretical framework for continuous monitoring of multi-particle quantum dynamics, and shown that quantum indistinguishability protects the system from the relative positional decoherence. Such a real-time observation can be applicable for realizing quantum feedback control of quantum many-body systems and future subwavelength-spacing lattices should be used for this purpose.

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formula of the diffusion constant.

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Supplemental Materials

Continuous limit of the position measurement of indistinguishable particles.

Here we discuss the continuous measurement limit of indistinguishable particles. In the continuous limit, time window becomes infinitesimal and the number of windows per unit time becomes infinite. It then follows from the central limit theorem that the Poisson stochastic process can be approximated by

$$dN(X) = \langle \hat{M}^\dagger(X) \hat{M}(X) \rangle dt + \sqrt{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle} dW(X), \quad (\text{S5})$$

where $dW(X)$ is the Wiener stochastic process subject to $dW(X)dW(Y) = \delta(X - Y)dt$ and $E[dW(X)] = 0$. By substituting this expression into Eq. (1) in the main text, we obtain

$$\begin{aligned} d\hat{\rho} &= \left[-i [\hat{H}, \hat{\rho}] - \frac{1}{2} \int dX \left\{ \hat{M}^\dagger(X) \hat{M}(X) - \langle \hat{M}^\dagger(X) \hat{M}(X) \rangle, \hat{\rho} \right\} \right] dt + \int dX \left(\frac{\hat{M}(X) \hat{\rho} \hat{M}^\dagger(X)}{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle} - \hat{\rho} \right) dN(X) \\ &= -i [\hat{H}, \hat{\rho}] dt - \frac{1}{2} \int dX [\hat{M}(X), [\hat{M}(X), \hat{\rho}]] dt + \int dX \left(\frac{\hat{M}(X) \hat{\rho} \hat{M}^\dagger(X)}{\sqrt{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle}} - \sqrt{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle} \hat{\rho} \right) dW(X). \end{aligned} \quad (\text{S6})$$

While we can take the continuous limit of Eq. (S6) for a general position measurement operator as discussed below, it is instructive to start with the derivation for the standard Gaussian position measurement. To do so, we first consider the measurement operator $\hat{M}(X) = \sqrt{\gamma/\sqrt{\pi\sigma^2}} \sum_m \exp[-(X - md)^2/(2\sigma^2)] \hat{n}_m$. In this case, the second term in Eq. (S6) can be explicitly calculated as

$$\begin{aligned} \frac{1}{2} \int dX [\hat{M}(X), [\hat{M}(X), \hat{\rho}]] &= \frac{\gamma}{2} \int dX \sum_{m,l} \frac{1}{\sqrt{\pi\sigma^2}} \exp \left[-\frac{(X - md)^2}{2\sigma^2} - \frac{(X - ld)^2}{2\sigma^2} \right] (\hat{n}_m \hat{n}_l \hat{\rho} + \hat{\rho} \hat{n}_l \hat{n}_m - 2\hat{n}_m \hat{\rho} \hat{n}_l) \\ &= \frac{\gamma}{2} \sum_{m,l} \exp \left[-\frac{(md - ld)^2}{4\sigma^2} \right] (\hat{n}_m \hat{n}_l \hat{\rho} + \hat{\rho} \hat{n}_l \hat{n}_m - 2\hat{n}_m \hat{\rho} \hat{n}_l) \\ &\simeq \frac{\gamma}{2} \sum_{m,l} \left(1 - \frac{(md - ld)^2}{4\sigma^2} \right) (\hat{n}_m \hat{n}_l \hat{\rho} + \hat{\rho} \hat{n}_l \hat{n}_m - 2\hat{n}_m \hat{\rho} \hat{n}_l) \\ &= -\frac{\gamma}{2} \sum_{m,l} \frac{(m^2 + l^2 - 2ml) d^2}{4\sigma^2} (\hat{n}_m \hat{n}_l \hat{\rho} + \hat{\rho} \hat{n}_l \hat{n}_m - 2\hat{n}_m \hat{\rho} \hat{n}_l) \\ &= \gamma \sum_{m,l} \frac{mld^2}{4\sigma^2} (\hat{n}_m \hat{n}_l \hat{\rho} + \hat{\rho} \hat{n}_l \hat{n}_m - 2\hat{n}_m \hat{\rho} \hat{n}_l) \\ &= \frac{N^2 \gamma d^2}{4\sigma^2} [\hat{X}_{CM}, [\hat{X}_{CM}, \hat{\rho}]]. \end{aligned} \quad (\text{S7})$$

From the second to the third line, we assume a situation in which the spatial resolution of the measurement is low so that interference peaks of atoms cannot be resolved. We use the relation of the particle conservation $\sum_m \hat{n}_m = N \hat{I}$ in deriving the fourth and fifth equalities, where N is the number of atoms and \hat{I} is the identity operator. We introduce the center-of-mass operator $\hat{X}_{CM} = \sum_m \hat{n}_m / N$ in the last line.

Next, let us consider the continuous limit of the last term in Eq. (S6). We note that a straightforward expansion of the exponential function of $\hat{M}(X)$ would result in a divergence due to an infinite range of the integration over X . To regulate this divergence, we first take the integration over a finite range $[-L, L]$, and after expanding the exponential

function, we take the large- L limit. Then, the third term can be calculated as

$$\begin{aligned}
& \int_{-L}^L dX \left(\frac{\hat{M}(X) \hat{\rho} \hat{M}^\dagger(X)}{\sqrt{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle}} - \sqrt{\langle \hat{M}^\dagger(X) \hat{M}(X) \rangle} \hat{\rho} \right) dW(X) \\
& \simeq \int_{-L}^L dX dW(X) \left[\frac{\gamma}{\sqrt{\pi\sigma^2}} \sum_m \left(1 - \frac{(X-md)^2}{2\sigma^2} \right) \hat{n}_m \hat{\rho} \sum_l \left(1 - \frac{(X-l d)^2}{2\sigma^2} \right) \hat{n}_l \right. \\
& \quad \left. - \sqrt{\frac{\gamma}{\sqrt{\pi\sigma^2}} \sum_{m',l'} \left(1 - \frac{(X-m'd)^2}{2\sigma^2} - \frac{(X-l'd)^2}{2\sigma^2} \right) \langle \hat{n}_{m'} \hat{n}_{l'} \rangle} \hat{\rho} \sqrt{\frac{\gamma}{\sqrt{\pi\sigma^2}} \sum_{m',l'} \left(1 - \frac{(X-m'd)^2}{2\sigma^2} - \frac{(X-l'd)^2}{2\sigma^2} \right) \langle \hat{n}_{m'} \hat{n}_{l'} \rangle} \right] \\
& \simeq \int_{-L}^L dX dW(X) \left[\sqrt{\frac{\gamma}{\sqrt{\pi\sigma^2}}} \left(\left(1 - \frac{X^2}{2\sigma^2} \right) N \hat{I} + \frac{NXd}{\sigma^2} \hat{X}_{CM} - \frac{Nd^2}{2\sigma^2} \hat{\Sigma} \right) \hat{\rho} \left(\left(1 - \frac{X^2}{2\sigma^2} \right) N \hat{I} + \frac{NXd}{\sigma^2} \hat{X}_{CM} - \frac{Nd^2}{2\sigma^2} \hat{\Sigma} \right) \right. \\
& \quad \times \frac{1}{N} \left(1 + \frac{X^2}{2\sigma^2} - \frac{Xd}{\sigma^2} \langle \hat{X}_{CM} \rangle + \frac{d^2}{2\sigma^2} \langle \hat{\Sigma} \rangle \right) - \hat{\rho} \sqrt{\frac{N^2\gamma}{\sqrt{\pi\sigma^2}}} \left(1 - \frac{X^2}{2\sigma^2} + \frac{Xd}{\sigma^2} \langle \hat{X}_{CM} \rangle - \frac{d^2}{2\sigma^2} \langle \hat{\Sigma} \rangle \right) \left. \right] \\
& \simeq \int_{-L}^L dX dW(X) \sqrt{\frac{N^2\gamma}{\sqrt{\pi\sigma^2}}} \left[\left(\frac{Xd}{\sigma^2} \hat{X}_{CM} - \frac{d^2}{2\sigma^2} \hat{\Sigma} \right) \hat{\rho} \left(\frac{Xd}{\sigma^2} \hat{X}_{CM} - \frac{d^2}{2\sigma^2} \hat{\Sigma} \right) - 2\hat{\rho} \left(\frac{Xd}{\sigma^2} \langle \hat{X}_{CM} \rangle - \frac{d^2}{2\sigma^2} \langle \hat{\Sigma} \rangle \right) \right], \tag{S8}
\end{aligned}$$

where we introduce the inertial moment operator $\hat{\Sigma} \equiv \sum_m m^2 \hat{n}_m / N$. To calculate the coefficient of each individual operator, we next use the fact that a linear superposition of Wiener stochastic processes is again a Wiener process aside from a constant factor:

$$\int_{-L}^L dX dW(X) = \sqrt{2L} dW, \quad \int_{-L}^L X dX dW(X) = \sqrt{\frac{2L^3}{3}} dW, \tag{S9}$$

where dW is the standard Wiener process satisfying $dW^2 = dt$ and $E[dW] = 0$. Substituting these relations into Eq. (S8), we obtain

$$\sqrt{\frac{N^2\gamma}{\sqrt{\pi\sigma^2}}} \left\{ \sqrt{\frac{2d^2L^3}{3\sigma^4}} \hat{X}_{CM} - \sqrt{\frac{d^4L}{2\sigma^4}} \hat{\Sigma} - \sqrt{\frac{2d^2L^3}{3\sigma^4}} \langle \hat{X}_{CM} \rangle + \sqrt{\frac{d^4L}{2\sigma^4}} \langle \hat{\Sigma} \rangle, \hat{\rho} \right\} dW. \tag{S10}$$

Since the coefficient of $\hat{\Sigma}$ is smaller than that of \hat{X}_{CM} by an order of d/L , we can neglect the contribution of the terms including $\hat{\Sigma}$ in Eq. (S10) in the continuous limit. Hence, the stochastic part becomes

$$\sqrt{\frac{2\gamma d^2 L^3}{3\sqrt{\pi}\sigma^5}} \left\{ \hat{X}_{CM} - \langle \hat{X}_{CM} \rangle, \hat{\rho} \right\} dW. \tag{S11}$$

To preserve the purity of the quantum state in the course of the stochastic time evolution, the coefficient of the stochastic part, Eq. (S11), should be consistent with that of the deterministic part, Eq. (S7). One can show that this uniquely determines the way of taking the limit of L such that the relation $L/\sigma = (3\sqrt{\pi}/4)^{1/3}$ is kept. Then, we finally obtain the continuous measurement limit of the position measurement of indistinguishable particles as

$$d\hat{\rho} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] dt - \frac{N^2\gamma d^2}{4\sigma^2} [\hat{X}_{CM}, [\hat{X}_{CM}, \hat{\rho}]] dt + \sqrt{\frac{N^2\gamma d^2}{2\sigma^2}} \left\{ \hat{X}_{CM} - \langle \hat{X}_{CM} \rangle, \hat{\rho} \right\} dW. \tag{S12}$$

Note that if we consider a single particle, Eq. (S12) reproduces the previous results for the continuous position measurement model of a single particle (see Refs. [35–39] in the main text).

A general point spread function

Let us discuss the case of a general measurement operator given as $\hat{M}(X) = \sqrt{\gamma} \sum_m [f_0((X-md)/\sigma_0)/\sqrt{\sigma_0}] \hat{n}_m$, where f_0 denotes a dimensionless amplitude of the point spread function and σ_0 is the bare resolution parameter. Note that the function f in the main text can be related to f_0 by $f(Z) = f_0(Z/\sigma_0)/\sqrt{\sigma_0}$. We assume that the function f_0 is properly normalized as $\int f_0^2(x) dx = 1$ and is a symmetric function centered at zero: $f_0(x) = f_0(-x)$.

Let us consider the following integral

$$\begin{aligned} K(m, l) &\equiv \int dX f(X - md) f(X - ld) \\ &= \int \frac{dX}{\sigma_0} f_0\left(\frac{X - md}{\sigma_0}\right) f_0\left(\frac{X - ld}{\sigma_0}\right). \end{aligned} \quad (\text{S13})$$

The integral K only depends on the difference between m and l , and is symmetric under the interchange of variables m and l . Hence, in the weak resolution limit $\sigma_0 \gg d$, the term linear in $(m - l)$ should vanish and the expansion of K begins with

$$K(m, l) \simeq 1 - \frac{(m - l)^2 d^2}{4\sigma_{\text{eff}}^2}, \quad (\text{S14})$$

where we use the normalization condition of the point spread function and introduce the effective resolution σ_{eff} as the expansion coefficient. In general, the effective parameter σ_{eff} should be the function of the bare resolution σ_0 . For example, for the Gaussian function $f_0(x) = e^{-x^2/2}/\pi^{1/4}$ discussed in the previous section, these values coincide: $\sigma_{\text{eff}} = \sigma_0 = \sigma$. Under these assumptions and the definition, one can derive the continuous limit for a general point spread function along the same line of the previous section. The deterministic part can be calculated in the same manner as demonstrated in the previous section. This determines the coefficient of the deterministic part as $N^2 \gamma d^2 / (4\sigma_{\text{eff}}^2)$. In taking the continuous limit of the stochastic part, the limit of the range of the integral should be taken so that the coefficient of the stochastic part is consistent with that of the deterministic part such that the purity of the quantum state in the time evolution is preserved. The final result becomes the same equation as Eq. (S12) with σ replaced by the effective resolution σ_{eff} . Note that we abbreviate σ_{eff} as σ in the main text for notational simplicity.

Derivation of the diffusion constant for distinguishable particles

We here derive an analytical formula of the diffusion constant of the relative distance of distinguishable particles. To this end, we first consider a single particle model. We denote $\langle n | \hat{\rho} | m \rangle = \rho_{n,m}$ where $|n\rangle$ is the state in which the particle localizes at site n . Then, the non-selective time evolution is described by

$$\dot{\rho}_{n,m} = iJ(\rho_{n+1,m} + \rho_{n-1,m} - \rho_{n,m+1} - \rho_{n,m-1}) - \frac{\Gamma}{4}(n - m)^2 \rho_{n,m}. \quad (\text{S15})$$

We are interested in the diffusive regime, i.e., in the long-time limit. In this regime, as can be seen from Eq. (S15), the coherence between remote different sites should be neglected because the decoherence rate becomes larger as the distance of different sites increases. Hence, we only consider the coherence between neighboring sites $\rho_{n,n\pm 1}$ and also assume that the time evolution of these values can be neglected $\dot{\rho}_{n,n\pm 1} \simeq 0$ in the stationary regime. This results in the following expression of the coherence:

$$\begin{aligned} \rho_{n,n\pm 1} &\simeq \frac{4iJ}{\Gamma}(\rho_{n\pm 1,n\pm 1} - \rho_{n,n}) \\ \rho_{n\pm 1,n} &\simeq -\frac{4iJ}{\Gamma}(\rho_{n\pm 1,n\pm 1} - \rho_{n,n}). \end{aligned} \quad (\text{S16})$$

By substituting these expressions into the time evolution equation of the on-site density component $\rho_{n,n}$, we obtain

$$\begin{aligned} \dot{\rho}_{n,n} &= iJ(\rho_{n+1,n} + \rho_{n-1,n} - \rho_{n,n+1} - \rho_{n,n-1}) \\ &\simeq \frac{8J^2}{\Gamma}(\rho_{n+1,n+1} - 2\rho_{n,n} + \rho_{n-1,n-1}), \end{aligned} \quad (\text{S17})$$

which is the diffusion equation with the diffusion constant $8J^2/\Gamma$. We note that similar discussions for the site-resolved decoherence model can be found in Refs. [53, 55] in the main text.

For non-interacting distinguishable particles, the variance of the relative distance between two particles is given as $\sigma_r^2 = \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle = \langle \hat{x}_1^2 \rangle + \langle \hat{x}_2^2 \rangle - 2\langle \hat{x}_1 \rangle \langle \hat{x}_2 \rangle$, where $\hat{x}_{1(2)}$ is the position operator of particle 1 (2). If particles are initially localized in single sites as considered in the main text, the expectation values of the positions $\langle \hat{x}_{1,2} \rangle$ should remain in the initial values. In particular, we choose the initial condition of two adjacent particles $\langle \hat{x}_1 \rangle_0 = 0$ and $\langle \hat{x}_2 \rangle_0 = 1$ in the numerical simulations of the main text. As a result, we obtain the long-time limit behavior of σ_r as

$$\sigma_r^2 = \frac{32J^2 t}{\Gamma} \equiv 2D_c t, \quad (\text{S18})$$

where we introduce the diffusion constant D_c for the relative distance.